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Jean Marc Vanel, Roger Peyret, Patrick Bontoux

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A PSEUDO-SPECTRAL SOLUTION OF VORTICITY-STREAM FUNCTION EQUATIONS USING THE INFLUENCE MATRIX TECHNIQUE

J.M. Vanel and R. Peyret
(Département de Mathématiques, Université de Nice, France)

and

P. Bontoux
(I.M.F.M., Université d'Aix-Marseille II, France)

1. INTRODUCTION

In this paper we present a fast and accurate method for computing unsteady two-dimensional incompressible flows. The method makes use of the vorticity and the stream function as dependent variables approximated by means of Chebyshev polynomial expansions. When using these variables the delicate question is the derivation of proper boundary conditions for the vorticity. The treatment of these conditions must preserve accuracy, stability and should avoid the use of an iterative solution procedure. The method developed here makes use of the influence matrix technique. This technique which has been found successful for the velocity-pressure equations (Kleiser and Schumann, 1980; Le Quéré and Alziary de Roquefort, 1982) has been applied until now to vorticity-stream function equations only in cases which reduce to one-dimensional problems (Tuckerman, 1983; Dennis and Quartapelle, 1983). Our purpose is to give a two-dimensional formulation of the method, to point out the theoretical and numerical difficulties associated with it, and finally to present numerical results illustrating the properties of the method.

2. EQUATIONS AND NUMERICAL APPROXIMATION

The Navier-Stokes equations within the vorticity-stream function formulation are

$$\frac{\partial \omega}{\partial t} - \frac{1}{Re} \nabla^2 \omega = - \vec{V} . \nabla \omega + f \equiv F(\psi, \omega; f) \qquad (2.1)$$

$$\nabla^2 \psi + \omega = 0 \qquad (2.2)$$

where the stream function ψ and the vorticity ω are connected to the velocity $\vec{V}$ by

$$\vec{V} = (u,v) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right), \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \qquad (2.3)$$

These equations are solved in D ($-1< x,y<1$) with the boundary conditions on $\Gamma = \partial D$.

$$\psi=g, \quad \frac{\partial\psi}{\partial\nu} = h \quad (\vec{\nu} = \text{unit normal to } \Gamma). \qquad (2.4a,b)$$

The initial condition at $t = 0$ is

$$\vec{V} = \vec{V}^{o}, \text{ then } \omega = \omega^{o}. \qquad (2.5)$$

In (2.1), Re is the Reynolds number and f is a given forcing term.

The time-discretization is a combination of the 2nd-order Backward Euler scheme for the viscous term with an Adams-Bashforth type evaluation of the convective term, i.e:

$$\frac{1}{2\Delta t}(3\omega^{n+1} - 4\omega^{n} + \omega^{n-1}) - \frac{1}{Re}\nabla^2\omega^{n+1} = 2F^{n} - F^{n-1} \qquad (2.6)$$

$$\nabla^2\psi^{n+1} + \omega^{n+1} = 0 \qquad (2.7)$$

where n refers to the time $t_n = n\Delta t$. This second order scheme is preferred to the usual Crank-Nicolson/Adams-Bashforth scheme for its better stability properties:

(1) the higher Fourier or Chebyshev modes are better damped,

(2) for moderate Re the critical time-step is larger, and for small Re the scheme is unconditionally stable, as found by Zakaria (1985) for the advection-diffusion equation solved with a Collocation-Chebyshev method.

So, at each time-step the following Stokes-type problem has to be solved:

$$\left.\begin{aligned} \nabla^2\omega - \lambda\omega &= G \qquad (2.8)\\ \nabla^2\psi + \omega &= 0 \qquad (2.9) \end{aligned}\right\} \text{ in D}$$

$$\psi = g, \quad \frac{\partial \psi}{\partial \nu} = h \text{ on } \Gamma \qquad (2.10a,b)$$

where ω and ψ are written for ω^{n+1} and ψ^{n+1}, respectively.

The approximation of (2.8) - (2.10) makes use of the Chebyshev polynomial expansion:

$$\begin{pmatrix} \omega \\ \psi \end{pmatrix} = \sum_{n=0}^{N} \sum_{m=0}^{M} \begin{pmatrix} \hat{\omega}_{n,m} \\ \hat{\psi}_{n,m} \end{pmatrix} T_n(x) T_m(y) \qquad (2.11)$$

The Chebyshev coefficients of G in (2.8) are calculated by the pseudo-spectral technique which consists in performing the products in the physical space and derivations in the spectral space, these two spaces being connected through a FFT algorithm.

The unknowns $\hat{\omega}_{n,m}$, $\hat{\psi}_{n,m}$ are obtained through the Tau-Method (see Gottlieb-Orszag, 1977) associated with the matrix diagonalization technique of Haidvogel and Zang (1979) for the solution of the systems.

The classical difficulty associated with the formulation lies in the determination of boundary conditions for the vorticity. The technique which consists in using $\omega = -\nabla^2\psi$ on the boundary has been employed in association with a Tau-Chebyshev method by Elie et al. (1983). We present here another solution to this problem which makes use of the influence matrix method. The advantage of this method lies in the strong coupling at the same time-level (n + 1) between ω and ψ on the boundary, leading to good accuracy in time and stability without requiring any iterative procedure.

3. INFLUENCE MATRIX METHOD

3.1 Description of the Method

Thanks to the linearity of the problem, the solution of (2.8)-(2.10) can be written as

$$\begin{pmatrix} \omega \\ \psi \end{pmatrix} = \begin{pmatrix} \tilde{\omega} \\ \tilde{\psi} \end{pmatrix} + \sum_{i=1}^{J} \gamma_i \begin{pmatrix} \omega_i \\ \psi_i \end{pmatrix} \qquad (3.1.1)$$

In this formula, $(\tilde{\omega},\tilde{\psi})$ is the solution of Problem A:

$$\begin{cases} \nabla^2\tilde{\omega}-\lambda\tilde{\omega} = G \text{ in } D \\ \tilde{\omega} = \tilde{\omega}_o \text{ (arbitrary) on } \Gamma \end{cases} \tag{3.1.2}$$

$$\begin{cases} \nabla^2\tilde{\psi} = -\tilde{\omega} \text{ in } D \\ \tilde{\psi} = g \quad \text{on } \Gamma \ . \end{cases} \tag{3.1.3}$$

In the summation in the equation (3.1.1) the number J is connected with the number of collocation points (ξ_j, η_j) on Γ. These (ω_i, ψ_i) are solutions of Problem B:

$$\begin{cases} \nabla^2\omega_i - \lambda\,\omega_i = 0 \quad \text{in } D \\ \omega_{ij} = \delta_{ij} \qquad \text{on } \Gamma \end{cases} \qquad \begin{cases} \nabla^2\psi_i = -\omega_i \quad \text{in } D \\ \psi_i = 0 \qquad \text{on } \Gamma \end{cases} \tag{3.1.4}$$

where j refers to the points (ξ_j,η_j) and δ_{ij} is the Kronecker symbol . Then the J coefficients γ_i are determined such that ψ satisfies the boundary conditions (2.10b), i.e.

$$\left(\frac{\partial\psi}{\partial\nu}\right)_j = h_j = \left(\frac{\partial\tilde{\psi}}{\partial\nu}\right)_j + \sum_{i=1}^{J} \gamma_i \left(\frac{\partial\psi_i}{\partial\nu}\right)_j \tag{3.1.5}$$

This equation, written for all j, yields an algebraic system for the γ_i's. The matrix $A = [\,(\partial\psi_i/\partial\nu)_j\,]$ of this system is the "influence matrix".

Note that: (1) Problem B is independent of the time-level considered. Consequently, the calculation of ω_i,ψ_i, A and its inverse A^{-1} can be done once and for all before starting the time-integration. (2) It is more economical to define the boundary values of ω

$$\omega_j = \tilde{\omega}_{oj} + \gamma_j \quad \text{on } \Gamma \tag{3.1.6}$$

and to solve a problem A [with (3.1.6) as boundary condition for ω] rather than to store the J fields (ω_i,ψ_i) and to perform the linear combination in (3.1.1).

3.2 Problem of corners and determination of J

The presence of corners in the domain D leads to difficulties. Two methods have been considered to surmount these.

Firstly, we assume that f,g and h are such that $\psi \in C^2$ so that ω at a corner C can be expressed by:

$$\omega_{|C} = - \nabla^2 \psi_{|C} \tag{3.2.1}$$

which is known from the condition (2.10a).

So, in Method 1, we prescribe

$$\tilde{\omega}_{|C} = \tilde{\omega}_{o|C} = - \nabla^2 \psi_{|C} \tag{3.2.2}$$

$$\omega_{i|C} = 0 \tag{3.2.3}$$

and the equation (3.1.5) is written at each collocation point on Γ except the corners; hence $J = 2(N + M - 2)$.

In Method 2, f,g and h are assumed such that $\nabla^4 \psi \in C^o$ so that the equation (2.8) is satisfied up to the corner C. Now, together with (3.2.2) and (3.2.3), we can prescribe

$$\nabla^2 \tilde{\omega}_{|C} = - \lambda \nabla^2 \psi_{|C} + G_{|C} \tag{3.2.4}$$

$$\nabla^2 \omega_{|C} = 0 \tag{3.2.5}$$

Because of these additional conditions, the equation (3.1.5) must be written at $J = 2(N + M - 4)$ collocation points only. At the four remaining points, the values of $\tilde{\omega} = \tilde{\omega}_o$ and ω_i are determined so that (3.2.4) and (3.2.5) are satisfied. Here, we have chosen these special points S adjacent to the corners (Fig. 1), that is to say in the regions where the density of collocation points is high.

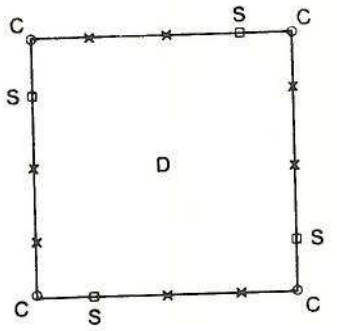

Fig. 1

3.3 Difficulty associated with the Tau-Method

The use of the Tau-Method to solve problem B for ψ_i leads to a difficulty arising because the last weighted residuals, corresponding to the higher modes, are not considered (they are replaced by the boundary conditions). So it appears that there exists a linear combination $\bar{\omega}$ of the ω_i such that the only non-zero coefficients have indices $n = N - 1$ or N, or $m = M - 1$ or M, precisely those coefficients which are not taken into account in the calculation of the corresponding stream function. Consequently we have a numerical solution $\bar{\psi} = 0$ and the matrix A is not invertible. This difficulty can be surmounted if the whole spectrum of the vorticity ω_i is considered when solving the equation for ψ_i. This is done by increasing the number of polynomials in the expansion (2.11) for ψ, which goes up to $N + 2$, $M + 2$ now.

We are thus led to the problem of expressing $\partial\psi/\partial x$ and $\partial\psi/\partial y$, defined by $(N + 3)(M + 3)$ coefficients, at the usual $(N + 1)(M + 1)$ collocation points $(\cos n\pi/N, \cos m\pi/M)$. The evaluation of G is done by simply neglecting the last two coefficients in either direction. On the boundary, two variants have been considered:

Method 1, 2: all the coefficients of $\partial\psi/\partial\nu$ are taken into account.
Method 1', 2': the two last modes of $\partial\psi/\partial\nu$ are neglected.

The numerical results have shown that 1 is much more accurate than 1', and that 2' is slightly better than 2.

4. NUMERICAL RESULTS

All the calculations reported here have been made with N = M.

4.1 Exact unsteady solution

We consider the solution

$$\psi_{ex} = \frac{2\pi - 1 + \sin 2\pi t}{2\pi}\ \psi_o,\ \psi_o = (1-x^2)^2(1-y^2)^2\ e^{x(y-1)} \tag{4.1.1}$$

which defines f, g and h. Here we compare the accuracy of various time-discretization schemes:

(1) 1^{st}-order Backward Euler/Adams-Bashforth

(2) 2^{nd}-order Backward Euler/Adams-Bashforth type [Eqs. (2.6) - (2.7)].

(3) Crank-Nicolson/Adams-Bashforth.

The calculations have been made with Re = 1, N = 20 and $0.1 \leq \Delta t \leq 0.005$. With these values of N and Δt, the leading part of the error comes from the time-discretization. Fig. 2 shows the normalized mean quadratic error:

$$\bar{E}_\omega = \underset{t}{\text{Max}}\{\|\omega - \omega_{ex}\|_{L2}/\|\omega_{ex}\|_{L2}\}\ . \tag{4.1.2}$$

The presence of three levels in time in the schemes necessitates starting-up procedures with modified schemes which, in the present calculations, are first order only. Hence, the maximum in (4.1.2) is taken when the effect of the initialization has disappeared: the error $\bar{E}_\omega$ is then periodic in time. From Fig. 2, it can be seen that scheme (2) is slightly more accurate than scheme (3), at least for the special solution computed here.

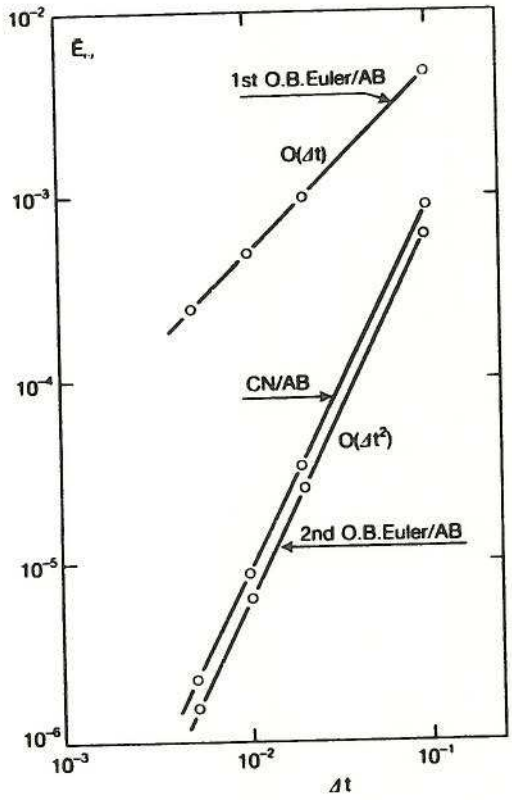

Fig. 2 Error $\bar{E}_\omega$

4.2 Exact steady solution

In order to compare the accuracy of methods 1 and 2' we have considered the steady solution $\psi_{ex} = \psi_o$, using the scheme (2.6) - (2.7) with Re = 1. From Table 1, it is seen that method 2' is more accurate than method 1 for small N. The accuracy is comparable with some advantage for Method 1 when N is larger.

N	10	12	16	20
Method 1	1.223×10^{-4}	5.828×10^{-6}	9.179×10^{-11}	4.045×10^{-11}
Method 2'	2.928×10^{-5}	3.507×10^{-7}	1.723×10^{-10}	3.573×10^{-10}

Table 1: Error $\bar{E}_\omega$

4.3 Steady Cavity flow

Finally, the method has been used to calculate the steady solution of (2.1) - (2.2) with f = 0, Re = 50 and with the boundary conditions: $\psi = 0$ on the whole boundary; $\partial\psi/\partial y = -(1 - x^2)^2$ at $y = 1$, $\partial\psi/\partial\nu = 0$ elsewhere; and with the initial condition $\vec{V}^0 = 0$ and therefore $\omega^0 = 0$.

Fig. 3, which shows the critical time-step, makes clear the good stability properties of the method.

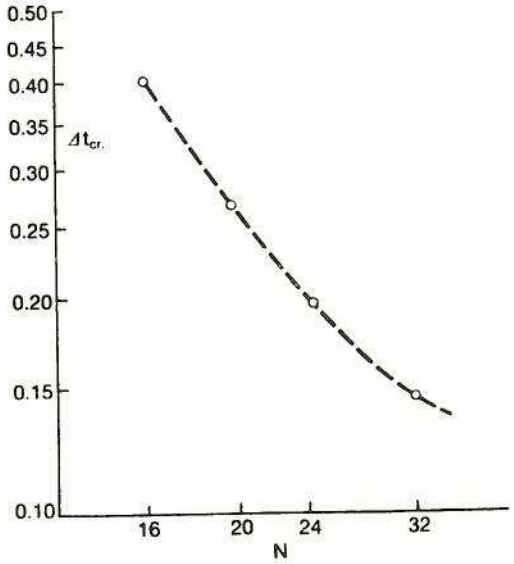

Fig. 3 Critical time-step

Table 2 gives some results. Here, the variables have been redefined by using the side L of the square as reference length, so that $0 \leq X, Y \leq 1, \Psi = \psi/2$. $\Omega = 2\omega$ and $Re_L = 100$. refer to (Peyret and Taylor, 1983) for results given by other methods. Because of the strong variation of Ω on the upper side $Y = 1$, the consideration of the maximum on collocation points only gives a bad idea of the convergence of the method with respect to N. Since the spectral approximation gives the solution at every $X \in [0,1]$, we have also listed in the last column of Table 2 the maximum value of $\Omega(X,1)$ on 201 equally spaced points.

Table 2. Results for cavity flow, $Re_L = 100$

Method	N	$Max\|\psi\|$ (e)	$Max\|\Omega(X,1)\|$ (e)	$Max\|\Omega(X,1)\|$ (f)
pseudo-spectral 1	16	0.083139	13.4235	13.5108
	20	0.082692	13.1519	13.4357
	24	0.083315	13.4149	13.4357
	32	0.083403	13.3418	13.4462
pseudo-spectral 2'	16	0.083158	13.3475	13.4527
	20	0.082695	13.1790	13.4506
	24	0.083315	13.4260	13.4490
	32	0.083403	13.3441	13.4441
pseudo-spectral (a)	16	0.083057	13.3574	13.4485
	20	0.082545	13.1810	13.4453
	24	0.083201	13.4315	13.4522
	32	0.083265	13.3496	13.4517
Collocation (V,p) (b)	16	0.083686	13.3574	13.4428
	32	0.083685	13.3422	13.4445
Hermitian (c)	20	0.0835	13.31	-
Finite-Diff. (d)	20	0.0829	13.14	-

(a) Elie et al. (1983),
(b) Ouazzani and Peyret (1983),
(c) 4th-order, Bontoux et al. (1978),
(d) 2nd-order, Peyret and Taylor (1983), p. 205, run 16,
(e) taken on collocation points,
(f) taken on 201 equally spaced points and obtained at X=0.62.

Fig. 4 shows the evolution of $\Omega(X,1)$. On the scale of the figure the difference between N = 16 and N = 32 cannot be distinguished. Fig. 5 is a blow-up of a very small region near X = 0. This provides an illustration of the high accuracy of Chebyshev approximation near the boundaries. Also, it is interesting to note the good agreement of the present results with those obtained by a Collocation - Chebyshev - Artificial Compressibility method (Ouazzani and Peyret, 1983) for the velocity-pressure equations, the latter method needing no special treatment at a corner.

The CPU time (CRAY 1S) is 0.03s/time-step for N = 20 and 0.078s/time-step for N = 32. The preprocessing is roughly equivalent to 4N time-steps.

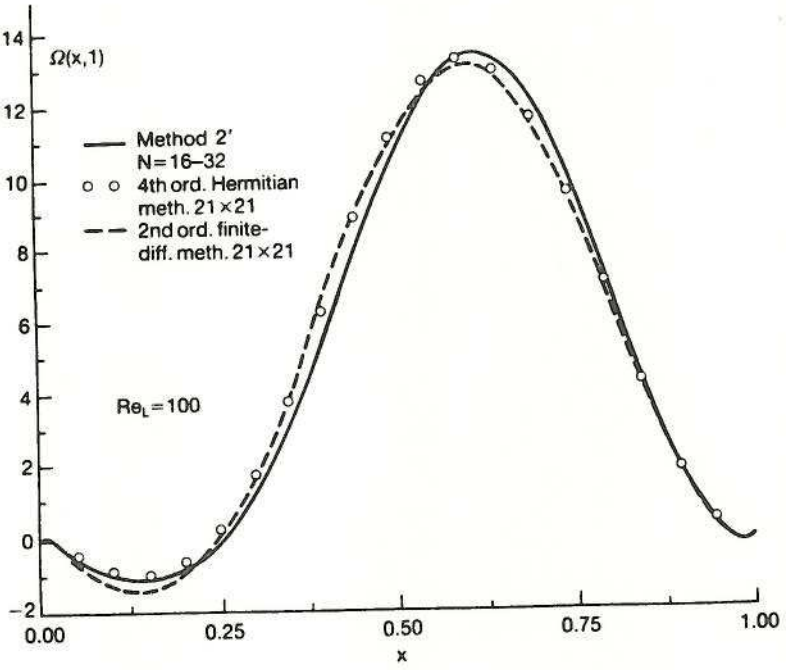

Fig. 4 Vorticity $\Omega(X, 1)$

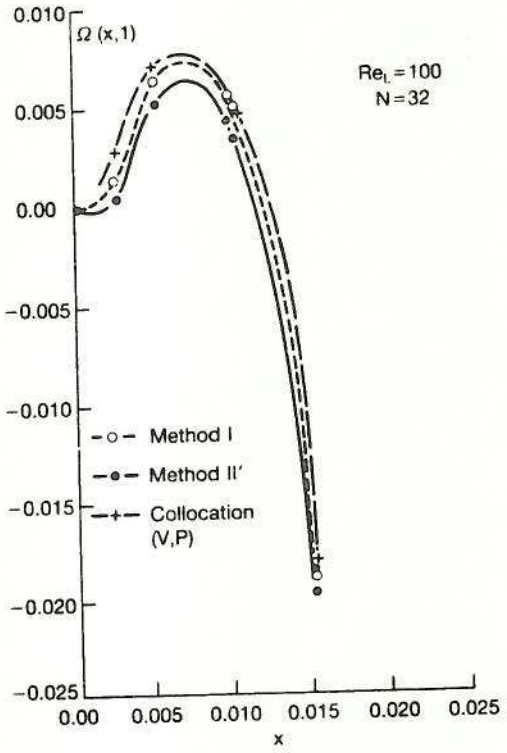

Fig. 5 Vorticity $\Omega(X, 1)$ near $X = 0$

5. ACKNOWLEDGEMENTS

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